Griffiths. 8.2

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{4E_c^2} \frac{|\vec{p}_3|}{|\vec{p}_1|},$$ ~~$\cos\theta$~~

$$|\vec{p}_3|^2 = \frac{E^2}{c^2} - M^2c^2, \quad |\vec{p}_1|^2 = \frac{E^2}{c^2} - m^2c^2$$

$$\frac{|\vec{p}_3|}{|\vec{p}_1|} = \sqrt{\frac{E^2/c^2 - M^2c^2}{E^2/c^2 - m^2c^2}}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{4E^2} \frac{\sqrt{E^2/c^2}\sqrt{1 - M^2c^4/E^2}}{\sqrt{E^2/c^2}\sqrt{1 - m^2c^4/E^2}}$$

This formula is θ-independent except $|\mathcal{M}|^2$, so we can independently integrate $|\mathcal{M}|^2$ over $d\Omega = \sin\theta d\theta d\phi$.

$$\langle|\mathcal{M}|^2\rangle = Q^2 g_e^4 \left\{1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2 + \left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)\right]\cos^2\theta\right\}.$$

Letting $A = \frac{mc^2}{E}$, $B = \frac{Mc^2}{E}$.

$$\langle|\mathcal{M}|^2\rangle = Q^2 g_e^4 \left\{1 + A^2 + B^2 + (1-A^2)(1-B^2)\cos^2\theta\right\}$$

$$\int \langle|\mathcal{M}|^2\rangle d\theta \sin\theta d\phi = Q^2 g_e^4 \int [1 + A^2 + B^2 + (1-A^2)(1-B^2)\cos^2\theta]\, d\Omega$$

$$= Q^2 g_e^4\, 4\pi \left[\int (1 + A^2 + B^2)\sin\theta d\theta + \int (1-A^2)(1-B^2)\cos^2\theta \sin\theta d\theta\right]$$

$$\int_0^{\pi}(1+A^2+B^2)\sin\theta\, d\theta = 1+A^2+B^2,$$

$$\int_0^{\pi}(1-A^2)(1-B^2)\cos^2\theta \sin\theta = \frac{1}{3}(1-A^2)(1-B^2)$$

$$1+A^2+B^2+\frac{1}{3}(1-A^2)(1-B^2) = \frac{4}{3}+\frac{2}{3}A^2+\frac{2}{3}B^2+\frac{1}{3}A^2B^2$$

$$= \frac{4}{3}\left[1+\frac{1}{2}A^2+\frac{1}{2}B^2+\frac{1}{4}A^2B^2\right]$$

$$= \frac{4}{3}\left[1+\frac{1}{2}A^2\right]\left[1+\frac{1}{2}B^2\right]$$

$$\Rightarrow \int \langle |M|^2 \rangle \, d\Omega = 4\pi Q^2 g_e^4 \frac{4}{3}\left[1+\frac{1}{2}A^2\right]\left[1+\frac{1}{2}B^2\right],$$

$$\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\, 4\pi Q^2 g_e^4}{\cancel{4} E_c^2}\frac{\cancel{4}}{3}\frac{\sqrt{1-B^2}}{\sqrt{1-A^2}}\left[1+\frac{1}{2}A^2\right]\left[1+\frac{1}{2}B^2\right]$$

$$\boxed{= \left(\frac{\hbar c}{8\pi E}\right)^2 \frac{S\, 4\pi}{3} Q^2 g_e^4 \frac{\sqrt{1-(Mc^2/E)^2}}{\sqrt{1-(mc^2/E)^2}}\left[1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]\left[1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right]}$$

moving proportionality factors into α shall give 8.5